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PERTURBATIONS IN THE MOTION OF ARTIFICIAL SATELLITES DUE TO THE ELLIPTICITY OF THE EARTH

by

V. F. Proskurin and Yu. V. Batrakov

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Literal expressions have been received for first order perturbations in the orbital elements of artificial satellites of the Earth accurate to the first power of the Earth's flattening and the fifth power of the orbital excentricity through. The coefficients of these expressions depend on the inclination of the orbit by means of trigonometrical polynomials. More accurate expressions are given for secular first order perturbations in the longitude of the node, in the argument of the perigee and in the mean anomaly. The secular motion of the node has been determined with account of second order perturbations due to the Earth's flattening. A numerical example illustrates the comparative value of the perturbations.
AUTHOR

One of the major causes of the orbital deviation of artificial satellites from the unperturbed Keplerian ellipses is the nonspherical shape of the Earth. And the greatest perturbations in the motion of the satellites are due to the ellipticity of the earth.

The problem of determining the motion of a satellite in the gravitational field of a flattened planet has been tackled before in connection with the development of the theories of motion of large planet satellites. But a number of orbital characteristics of the artificial satellites make it impossible to utilize the available methods for the development of a theory of their motion. Chief among them are the orbital inclination of the artificial satellites and their proximity to the Earth's surface.

It is therefore necessary to develop a new analytic theory that could be applied to artificial satellites with any orbital inclination toward the equatorial plane, and would be sufficiently accurate even in the case of satellites traveling in the immediate vicinity of the Earth's surface.

It is now assumed that the planet is shaped like an oblate[?] [Urovennyy]

 [redacted]

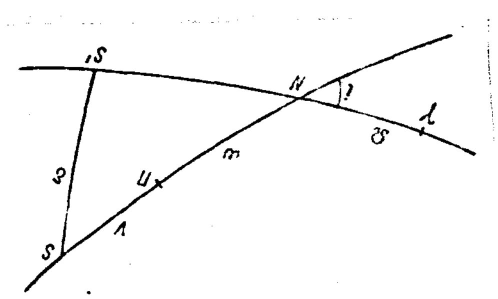
ellipsoid of revolution, and that its flattening is so insignificant that its third power may be disregarded. The part of the perturbation function under consideration is expanded in series according to the degree of eccentricity; the coefficients of the series are the trigonometric functions of inclination. The integration of the Lagrange equations produced analytical expressions for perturbations of the first order in relation to the compression of all the orbital elements, accurate to the fifth power of eccentricity inclusive. Purely secular terms of the motion of a node in the satellite orbit have been obtained from the perturbations of the second order in relation to the oblateness.

1. Formulating the Problem. Expanding the Perturbation Function

Let a zero mass satellite (S in the figure) move in the gravitational field of the Earth whose surface is shaped like an ellipsoid of revolution. The oblateness of the Earth and its angular velocity may be considered as insignificant magnitudes. The resistance of the atmosphere is not taken into consideration.

The potential of the oblate[?][Urovennyy] ellipsoid of revolution on an external point, accurate to the second power of oblateness, looks like the following

$$V = \frac{fm}{r}\left[1 - \frac{J}{3}\left(\frac{a'}{r}\right)^2 (3\sin^2\delta - 1) + \frac{D}{35}\left(\frac{a'}{r}\right)^4 (35\sin^4\delta - 30\sin^2\delta + 3)\right] = \frac{fm}{r} + R, \qquad (1)$$



where f is the gravitation constant, m the mass of the Earth, r the radius-vector of point S, a' the equatorial radius of the Earth, δ the deviation of the satellite. The coefficients J and D are defined by the following formulas

$$\left.\begin{aligned} J &= \varepsilon - \frac{m}{2} + \varepsilon\left(\frac{\varepsilon}{2} - \frac{m}{7}\right), \\ D &= \frac{\varepsilon}{4}(5J + 9\varepsilon), \\ m &= \frac{\omega^2 a'^3}{fm}(1 - \varepsilon), \end{aligned}\right\} \tag{2}$$

where ε is the flattening of the earth ellipsoid, and ω the angular speed of its revolution.

If perturbation function R, which is defined by formula (1), is limited to the first power of ellipticity, then

$$R = -\frac{1}{3} J \frac{fma'^2}{r^3}(3\sin^2\delta - 1). \tag{3}$$

To expand function R into series according to the degree of eccentricity of the satellite's orbit, it would be practical to begin by transforming expression (3). From the spherical triangle SNS' we find that

$$\sin\delta = \sin i \sin(v + \omega), \tag{4}$$

where i is the orbital incline, v the true anomaly, ω the angular distance from perigee to the node (perigee argument). Substituting (4) in (3) and carrying out some transformations, we get

$$R = \frac{1}{6} Jfm \frac{a'^2}{a^3}\left[(2 - 3\lambda^2)\left(\frac{a}{r}\right)^3 + 3\lambda^2 \cos 2\omega \left(\frac{a}{r}\right)^3 \cos 2v - 3\lambda^2 \sin 2\omega \left(\frac{a}{r}\right)^3 \sin 2v\right], \tag{5}$$

where a is the large orbital semiaxis, and $\lambda = \sin i$ for purposes of brevity. The expansion in series based on the multiples of mean anomoly M (Subbotin, 1937) can be used for the following combinations $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3 \cos 2v$, $\left(\frac{a}{r}\right)^3 \sin 2v$. We will write these expansions out to the sixth power of eccentricity e, inclusive.

$$
\left.\begin{aligned}
\left(\frac{a}{r}\right)^3 &= 1 + \frac{3}{2} e^2 + \frac{15}{8} e^4 + \frac{35}{16} e^6 + \left(3e + \frac{27}{8} e^3 + \frac{261}{64} e^5\right) \cos M + \\
&+ \left(\frac{9}{2} e^2 + \frac{7}{2} e^4 + \frac{141}{32} e^6\right) \cos 2M + \left(\frac{53}{8} e^3 + \frac{392}{128} e^5\right) \cos 3M + \\
&+ \frac{7}{8} e^4 + \frac{129}{80} e^6) \cos 4M + \frac{1773}{128} e^5 \cos 5M + \frac{3167}{160} e^6 \cos 6M, \\
\left(\frac{a}{r}\right) \cos 2v &= \left(-\frac{1}{2} e + \frac{1}{12} e^3 + \frac{1}{768} e^5\right) \cos M + \Big(1 - \frac{5}{2} e^2 + \\
&+ \frac{41}{48} e^4 - \frac{133}{1440} e^6\Big) \cos 2M + \left(\frac{7}{2} e - \frac{123}{16} e^3 + \frac{4971}{1280} e^5\right) \cos 3M + \\
&+ \left(\frac{17}{2} e^2 - \frac{115}{6} e^4 + \frac{9079}{720} e^6\right) \cos 4M + \left(\frac{845}{48} e^3 + \frac{32\,525}{768} e^5\right) \cos 5M + \\
&+ \left(\frac{533}{16} e^4 - \frac{13\,827}{160} e^6\right) \cos 6M + \frac{228\,347}{3840} e^5 \cos 7M + \frac{73\,369}{720} e^6 \cos 8M, \\
\left(\frac{a}{r}\right)^3 \sin 2v &= \left(-\frac{1}{2} e + \frac{1}{24} e^3 - \frac{7}{256} e^5\right) \sin M + \Big(1 - \frac{5}{2} e^2 + \\
&+ \frac{37}{48} e^4 - \frac{217}{1440} e^6\Big) \sin 2M + \left(\frac{7}{2} e - \frac{123}{16} e^3 + \frac{4809}{1280} e^5\right) \sin 3M + \\
&+ \left(\frac{17}{2} e^2 - \frac{115}{6} e^4 + \frac{3951}{720} e^6\right) \sin 4M + \left(\frac{845}{48} e^3 - \frac{32\,525}{768} e^5\right) \sin 5M + \\
&+ \left(\frac{533}{16} e^4 - \frac{13\,827}{160} e^6\right) \sin 6M + \frac{228\,347}{3840} e^5 \sin 7M + \frac{73\,369}{720} e^6 \sin 8M.
\end{aligned}\right\} \quad (6)
$$

Substituting (6) in (3), we get the final expression for R, correct to the sixth power of eccentricity

$$
\begin{aligned}
R = \frac{1}{3} Jfm \frac{a'^2}{a^3} \left(1 - \frac{3}{2} \lambda^2\right) \Big[1 &+ \frac{3}{2} e^2 + \frac{15}{8} e^4 + \frac{35}{16} e^6 + 3\left(e + \frac{9}{8} e^3 + \frac{87}{64} e^5\right) \cos M + \\
&+ \frac{9}{2}\left(e^2 + \frac{7}{9} e^4 + \frac{47}{48} e^6\right) \cos 2M + \frac{53}{8}\left(e^3 + \frac{393}{848} e^5\right) \cos 3M + \frac{77}{8}\left(e^4 + \frac{129}{770} e^6\right) \cos 4M + \\
&+ \frac{1773}{128} e^5 \cos 5M + \frac{3167}{160} e^6 \cos 6M \Big] + \frac{1}{2} Jfm \frac{a'^2}{a^3} \lambda^2 \Big[\frac{1}{48}\left(e^3 + \frac{11}{16} e^5\right) \cos (M - 2\omega) + \\
&+ \frac{1}{24}\left(e^4 + \frac{7}{10} e^6\right) \cos (2M - 2\omega) + \frac{81}{1280} e^5 \cos (3M - 2\omega) + \frac{4}{45} e^6 \cos (4M - 2\omega) - \\
&- \frac{1}{2}\left(e - \frac{1}{8} e^3 + \frac{5}{192} e^5\right) \cos (M + 2\omega) + \left(1 - \frac{5}{2} e^2 + \frac{13}{16} e^4 - \frac{35}{288} e^6\right) \cos (2M + 2\omega) + \\
&+ \frac{7}{2}\left(e - \frac{123}{56} e^3 + \frac{489}{448} e^5\right) \cos (3M + 2\omega) + \frac{17}{2}\left(e^2 - \frac{115}{51} e^4 + \frac{601}{408} e^6\right) \cos (4M + 2\omega) + \\
&+ \frac{845}{48}\left(e^3 - \frac{6505}{2704} e^5\right) \cos (5M + 2\omega) + \frac{533}{16}\left(e^4 - \frac{13\,827}{5330} e^6\right) \cos (6M + 2\omega) + \\
&+ \frac{228\,347}{3840} e^5 \cos (7M + 2\omega) + \frac{73\,369}{720} e^6 \cos (8M + 2\omega)\Big].
\end{aligned} \quad (7)
$$

2. Perturbations of the First Order

The Lagrange formulas used to define the osculating elements look like the following (Subbotin, 1937)

$$
\left.
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \varepsilon}, \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}\cdot\frac{\partial R}{\partial \pi} - \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cdot\frac{1}{na^2}\cdot\frac{\partial R}{\partial \varepsilon}, \\
\frac{di}{dt} &= -\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial \Omega} - \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \pi}+\frac{\partial R}{\partial \varepsilon}\right), \\
\frac{d\Omega}{dt} &= \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i}, \\
\frac{d\pi}{dt} &= \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\cdot\frac{\partial R}{\partial e}, \\
\frac{\partial \varepsilon}{\partial t} &= -\frac{2}{na}\cdot\frac{\partial R}{\partial a} + \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\cdot\frac{\partial R}{\partial i} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cdot\frac{1}{na^2}\cdot\frac{\partial R}{\partial e},
\end{aligned}
\right\}
\tag{8}
$$

where a, e, i, Ω, $\pi=\omega+\Omega$, ε represent six elements of the satellite orbit, n the average motion, and R the perturbation function.

We should point out that in the equation for π , the derivative $\frac{\partial R}{\partial e}$ is also raised to a negative e power. This reduces the accuracy of the expression of the first order in element π , so that the expression $e^{\delta_1\pi}$, required for the definition of the perturbed radius-vector and longitude of the satellite is accurate only to the fifth power of eccentricity. The perturbations and other elements should be defined with such accuracy, even though they could be determined to the sixth power of eccentricity inclusive.

Expanding the coefficients of the right parts of (8) by degree of eccentricity and substituting R from (7), we will get, after integrating (8), the following expressions for the first-order perturbations of the elliptic elements in the satellite orbit:

$$\frac{\delta_1 a}{a} = 2J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[\left(e+\frac{9}{8}e^3+\frac{87}{64}e^5\right)\cos M+\frac{3}{2}\left(e^2+\frac{7}{9}e^4\right)\cos 2M+$$
$$+\frac{53}{24}\left(e^3+\frac{393}{848}e^5\right)\cos 3M+\frac{77}{24}e^4\cos 4M+\frac{591}{128}e^5\cos 5M\Big]+$$
$$+J\left(\frac{a'}{a}\right)^2\lambda^2\Big[-\frac{1}{2}\left(e-\frac{1}{8}e^3+\frac{5}{192}e^5\right)\cos(M+2\omega)+\frac{1}{48}\left(e^3+\frac{11}{16}e^5\right)\cos(M-2\omega)+$$
$$+\left(1-\frac{5}{2}e^2+\frac{13}{16}e^4\right)\cos(2M+2\omega)+\frac{1}{24}e^4\cos(2M-2\omega)+$$
$$+\frac{7}{2}\left(e-\frac{123}{56}e^3+\frac{489}{448}e^5\right)\cos(3M+2\omega)+\frac{81}{1280}e^5\cos(3M-2\omega)+$$
$$+\frac{17}{2}\left(e^2-\frac{115}{51}e^4\right)\cos(4M+2\omega)+\frac{845}{48}\left(e^3-\frac{6505}{2704}e^5\right)\cos(5M+2\omega)+$$
$$+\frac{533}{16}e^4\cos(6M+2\omega)+\frac{228\,347}{3840}e^5\cos(7M+2\omega)\Big]. \tag{9a}$$

$$\delta_1 e = J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[\left(1+\frac{1}{8}e^2+\frac{15}{64}e^4\right)\cos M+\frac{3}{2}\left(e-\frac{2}{9}e^3+\frac{29}{144}e^5\right)\cos 2M+$$
$$+\frac{53}{24}\left(e^2-\frac{455}{848}e^4\right)\cos 3M+\frac{77}{24}\left(e^3-\frac{641}{770}e^5\right)\cos 4M+\frac{591}{128}e^4\cos 5M+\frac{3167}{480}e^5\cos 6M\Big]+$$
$$+\frac{1}{2}J\left(\frac{a'}{a}\right)^2\lambda^2\Big[\frac{1}{2}\left(1-\frac{1}{8}e^2-\frac{43}{192}e^4\right)\cos(M+2\omega)+\frac{1}{16}\left(e^2+\frac{1}{48}e^4\right)\cos(M-2\omega)+$$
$$+\frac{1}{12}\left(e^3-\frac{1}{20}e^5\right)\cos(2M-2\omega)-\frac{1}{2}\left(e-\frac{11}{4}e^3+\frac{21}{16}e^5\right)\cos(2M+2\omega)+$$
$$+\frac{7}{6}\left(1-\frac{235}{56}e^2+\frac{2569}{448}e^4\right)\cos(3M+2\omega)+\frac{27}{256}e^4\cos(3M-2\omega)+$$
$$+\frac{17}{4}\left(e-\frac{383}{102}e^3+\frac{256}{51}e^5\right)\cos(4M+2\omega)+\frac{2}{15}e^5\cos(4M-2\omega)+$$
$$\frac{169}{16}\left(e^2-\frac{30\,331}{8112}e^4\right)\cos(5M+2\omega)+\frac{533}{24}\left(\quad-\frac{40\,979}{10\,660}e^5\right)\cos(6M+2\omega)+$$
$$+\frac{32\,621}{768}e^4\cos(7M+2\omega)+\frac{73\,369}{960}e^5\cos(8M+2\omega)\Big], \tag{9b}$$

$$\delta_1 i = \frac{1}{2}J\left(\frac{a'}{a}\right)^2\lambda\sqrt{1-\lambda^2}\Big[-\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\cos(M+2\omega)-\frac{1}{24}\left(e^3+\frac{19}{16}e^5\right)\cos(M-2\omega)+$$
$$+\left(1-2e^2-\frac{1}{16}e^4\right)\cos(2M+2\omega)-\frac{1}{24}e^4\cos(2M-2\omega)+$$
$$+\frac{7}{3}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\cos(3M+2\omega)-\frac{27}{640}e^5\cos(3M-2\omega)+$$
$$+\frac{17}{4}\left(e^2-\frac{179}{102}e^4\right)\cos(4M+2\omega)+\frac{169}{24}\left(e^3-\frac{5153}{2704}e^5\right)\cos(5M+2\omega)+$$
$$+\frac{533}{48}e^4\cos(6M+2\omega)+\frac{32\,621}{1920}e^5\cos(7M+2\omega)\Big], \tag{9c}$$

$$
\begin{aligned}
\delta_1\Omega = & -J\left(\frac{a'}{a}\right)^2 \sqrt{1-\lambda^2}\Big[(1+2e^2+3e^4)\,nt+3\left(e+\frac{13}{8}e^3+\frac{147}{64}e^5\right)\sin M+\\
& +\frac{9}{4}\left(e^2+\frac{23}{18}e^4\right)\sin 2M+\frac{53}{24}\left(e^3+\frac{817}{848}e^5\right)\sin 3M+\frac{77}{32}e^4\sin 4M+\frac{1773}{640}e^5\sin 5M+\\
& +\frac{1}{2}\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\sin(M+2\omega)-\frac{1}{48}\left(e^3+\frac{19}{16}e^5\right)\sin(M-2\omega)-\\
& -\frac{1}{2}\left(1-2e^2-\frac{1}{16}e^4\right)\sin(2M+2\omega)-\frac{1}{48}e^4\sin(2M-2\omega)-\\
& -\frac{7}{6}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\sin(3M+2\omega)-\frac{27}{1280}e^5\sin(3M-2\omega)-\\
& -\frac{17}{8}\left(e^2-\frac{179}{102}e^4\right)\sin(4M+2\omega)-\frac{169}{48}\left(e^3-\frac{5153}{2704}e^5\right)\sin(5M+2\omega)-\\
& -\frac{533}{96}e^4\sin(6M+2\omega)-\frac{32\,621}{3840}e^5\sin(7M+2\omega)\Big],
\end{aligned}
\tag{9d}
$$

$$
\begin{aligned}
e\delta_1\pi = & -J\left(\frac{a'}{a}\right)^2(\sqrt{1-\lambda^2}-1+\lambda^2)\Big[(e+2e^3+3e^5)\,nt+3\left(e^2+\frac{13}{8}e^4\right)\sin M+\\
& +\frac{9}{4}\left(e^3+\frac{23}{18}e^5\right)\sin 2M+\frac{53}{24}e^4\sin 3M+\frac{77}{32}e^5\sin 4M+\frac{1}{2}\left(e^2+\frac{3}{8}e^4\right)\sin(M+2\omega)-\\
& -\frac{1}{48}e^4\sin(M-2\omega)-\frac{1}{2}\left(e-2e^3-\frac{1}{16}e^5\right)\sin(2M+2\omega)-\frac{1}{48}e^5\sin(2M-2\omega)-\\
& -\frac{7}{6}\left(e^2-\frac{95}{56}e^4\right)\sin(3M+2\omega)-\frac{17}{8}\left(e^3-\frac{179}{102}e^5\right)\sin(4M+2\omega)-\frac{169}{48}e^4\sin(5M+2\omega)-\\
& -\frac{533}{96}e^5\sin(6M+2\omega)\Big]+J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Big[(e+2e^3+3e^5)\,nt+\left(1+\frac{23}{8}e^2+\frac{319}{64}e^4\right)\sin M+\\
& +\frac{3}{2}\left(e+\frac{19}{18}e^3+\frac{293}{141}e^5\right)\sin 2M+\frac{53}{24}\left(e^2+\frac{231}{848}e^4\right)\sin 3M+\frac{77}{24}\left(e^3-\frac{383}{1540}e^5\right)\sin 4M+\\
& +\frac{591}{128}e^4\sin 5M+\frac{3167}{480}e^5\sin 6M\Big]-\frac{1}{4}J\left(\frac{a'}{a}\right)^2\lambda^2\Big[\left(1-\frac{7}{8}e^2+\frac{37}{192}e^4\right)\sin(M+2\omega)-\\
& -\frac{1}{8}\left(e^2+\frac{31}{48}e^4\right)\sin(M-2\omega)+5\left(e-\frac{23}{20}e^3+\frac{83}{240}e^5\right)\sin(2M+2\omega)-\\
& -\frac{1}{6}\left(e^3+\frac{11}{20}e^5\right)\sin(2M-2\omega)-\frac{27}{128}e^4\sin(3M-2\omega)-\\
& -\frac{7}{3}\left(1-\frac{397}{56}e^2+\frac{3865}{448}e^4\right)\sin(3M+2\omega)-\frac{4}{15}e^5\sin(4M-2\omega)-\\
& -\frac{17}{2}\left(e-\frac{511}{102}e^3+\frac{334}{51}e^5\right)\sin(4M+2\omega)-\frac{169}{8}\left(e^2-\frac{36\,581}{8112}e^4\right)\sin(5M+2\omega)-\\
& -\frac{533}{12}\left(e^3-\frac{46\,811}{10\,660}e^5\right)\sin(6M+2\omega)-\frac{32\,621}{384}e^4\sin(7M+2\omega)-\frac{73\,369}{480}e^5\sin(8M+2\omega)\Big],
\end{aligned}
\tag{9e}
$$

$$
\begin{aligned}
\delta_{1}\varepsilon = {} & 2J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\lambda^2\right)\Bigg[\left(1+\frac{7}{4}e^2+\frac{39}{16}e^4\right)nt+\frac{13}{4}\left(e+\frac{133}{104}e^3+\frac{109}{64}e^5\right)\sin M+ \\
& +\frac{21}{8}\left(e^2+\frac{215}{522}e^4\right)\sin 2M+\frac{265}{96}\left(e^3+\frac{403}{848}e^5\right)\sin 3M+\frac{77}{24}e^4\sin 4M+\frac{10\,047}{2560}e^5\sin 5M\Bigg]+ \\
& +3J\left(\frac{a'}{a}\right)^2\lambda^2\Bigg[-\frac{13}{24}\left(e-\frac{17}{104}e^3+\frac{79}{2496}e^5\right)\sin(M+2\omega)+\frac{5}{192}\left(e^3+\frac{35}{48}e^5\right)\sin(M-2\omega)+ \\
& +\frac{1}{2}\left(1-\frac{35}{12}e^2+\frac{19}{16}e^4\right)\sin(2M+2\omega)+\frac{1}{36}e^4\sin(2M-2\omega)+ \\
& +\frac{91}{72}\left(e-\frac{143}{56}e^3+\frac{8995}{5824}e^5\right)\sin(3M+2\omega)+\frac{153}{5120}e^5\sin(3M-2\omega)+ \\
& +\frac{119}{48}\left(e^2-\frac{533}{204}e^4\right)\sin(4M+2\omega)+\frac{845}{192}\left(e^3-\frac{112\,613}{40\,560}e^5\right)\sin(5M+2\omega)+ \\
& +\frac{533}{72}e^4\sin(6M+2\omega)+\frac{554\,557}{46\,080}e^5\sin(7M+2\omega)\Bigg]- \\
& -J\left(\frac{a'}{a}\right)^2\left(\sqrt{1-\lambda^2}-1+\lambda^2\right)\Bigg[(1+2e^2+3e^4)\,nt+3\left(e+\frac{13}{8}e^3+\frac{147}{64}e^5\right)\sin M+ \\
& +\frac{9}{4}\left(e^2+\frac{23}{18}e^4\right)\sin 2M+\frac{53}{24}\left(e^3+\frac{817}{848}e^5\right)\sin 3M+\frac{77}{32}e^4\sin 4M+\frac{1773}{640}e^5\sin 5M+ \\
& +\frac{1}{2}\left(e+\frac{3}{8}e^3+\frac{65}{192}e^5\right)\sin(M+2\omega)-\frac{1}{48}\left(e^3+\frac{19}{16}e^5\right)\sin(M-2\omega)- \\
& -\frac{1}{2}\left(1-2e^2-\frac{1}{16}e^4\right)\sin(2M+2\omega)-\frac{1}{48}e^4\sin(2M-2\omega)- \\
& -\frac{7}{6}\left(e-\frac{95}{56}e^3+\frac{165}{448}e^5\right)\sin(3M+2\omega)-\frac{27}{1280}e^5\sin(3M-2\omega)- \\
& -\frac{17}{8}\left(e^2-\frac{179}{102}e^4\right)\sin(4M+2\omega)-\frac{169}{48}\left(e^3-\frac{5153}{2704}e^5\right)\sin(5M+2\omega)-\frac{533}{96}e^4\sin(6M+2\omega)- \\
& -\frac{32\,521}{3840}e^5\sin(7M+2\omega)\Bigg]. \qquad (9f)
\end{aligned}
$$

The resulting perturbations of the elements can be used for computing the perturbed states of the artificial satellites. The following formulas are used for that purpose

$$
\begin{gathered}
M = M_0 + n_0(t-t_0) + \delta_{1}\varepsilon - \delta_{1}\pi - \frac{3}{2}\frac{n_0}{a_0}\int_{t_0}^{t}\delta_{1}a\,dt, \\
E = M + e\sin E, \\
r = a(1-e\cos E), \\
\operatorname{tg}\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}, \\
u = v + \omega, \\
x = r(\cos u\cos\Omega - \sin u\sin\Omega\cos i), \\
y = r(\cos u\sin\Omega + \sin u\cos\Omega\cos i), \\
z = r\sin u\sin i,
\end{gathered}
$$

where the letters with zero indices indicate the values of the unperturbed elements. We should point out the secular perturbations are found only in the elements Ω, π, ε. They are absent in the other elements.

3. Secular Perturbations of the First Order

The most important among the perturbations of the first order are the secular perturbations, as they determine the evolution of the orbit in the course of time. It is therefore useful to know the secular disturbances to a higher degree of accuracy.

The special structure of the perturbation function makes it possible to find the secular disturbances in their final form without resorting to an expansion in series based on the degree of eccentricity.

The computation of the secular disturbances can be more conveniently done by replacing the elements π, ε with ω and M_0 which are defined by the following formulas

$$\omega = \pi - \Omega, \quad M_0 = \varepsilon - \pi.$$

The Lagrange equations for the elements ω, M_0, look like the following

$$\left.\begin{aligned} \frac{d\omega}{dt} &= -\frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2e}\cdot\frac{\partial R}{\partial e}, \\ \frac{dM_0}{dt} &= -\frac{1-e^2}{na^2e}\cdot\frac{\partial R}{\partial e} - \frac{2}{na}\cdot\frac{\partial R}{\partial a}. \end{aligned}\right\} \tag{10}$$

We will designate the coefficients of the secular disturbances of elements Ω, ω, M_0 by Ω', ω', M_0', respectively. We will also introduce, as an independent variable, the true anomaly which is connected with t in the following formula $$r^2 dv = \sqrt{fm}\sqrt{a(1-e^2)}\,dt.$$

We will then have

$$\left.\begin{aligned} \omega' &= \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{d\omega}{dt}\,dv, \\ \Omega' &= \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{d\Omega}{dt}\,dv, \\ M_0' &= \frac{1}{2\pi\sqrt{fm}\sqrt{a(1-e^2)}}\int_0^{2\pi} r^2\frac{dM_0}{dt}\,dv. \end{aligned}\right\} \tag{11}$$

Formulas (11) produce the coefficients Ω', ω', M_o' on the assumption that the $\frac{T}{2\pi}$ magnitude has been selected as a unit of time, where T is the period of the satellite's revolution around the Earth. If a 24-hour day is to be taken as a unit of time, the right-hand parts of formulas (11) should be multiplied by the average daily motion of satellite n, and if the latter is expressed in radians or degrees, the resulting coefficients Ω', ω', M_o' will also be expressed in radians or degrees.

The calculation by formulas (11), following the multiplication of the right-hand parts by n, will produce[1]

$$\left.\begin{aligned} \Omega' &= -J\left(\frac{a'}{a}\right)^2 \frac{\cos i}{(1-e^2)^2} n, \\ \omega' &= \frac{1}{2} J\left(\frac{a'}{a}\right)^2 \frac{5\cos^2 i - 1}{(1-e^2)^2} n, \\ M_0' &= \frac{1}{2} J\left(\frac{a'}{a}\right)^2 \frac{3\cos^2 i - 1}{(1-e^2)^{3/2}} n. \end{aligned}\right\} \tag{12}$$

From the above it will be easy to form expressions for the coefficients of the secular perturbations π and ε. We have

$$\left.\begin{aligned} \pi' &= \frac{1}{2} J\left(\frac{a'}{a}\right)^2 \frac{5\cos^2 i - 2\cos i - 1}{(1-e^2)^2} n, \\ \varepsilon' &= \frac{1}{2} J\left(\frac{a'}{a}\right)^2 \frac{\left(5+3\sqrt{1-e^2}\right)\cos^2 i - 2\cos i - 1 - \sqrt{1-e^2}}{(1-e^2)^2} n. \end{aligned}\right\} \tag{13}$$

We can see from (12) that the secular motion of a node reaches its maximum at $i = 0, 180^\circ$, that is when the orbit lies in the equatorial plane. At $i = 90^\circ$, that is in case the orbital plane crosses the Earth's pole, $\Omega' = 0$. The secular motion of the perigee reaches a maximum at $i = 0, 180^\circ$ and is reduced to zero at $i = 63^\circ 26'$. The secular motion of the element M_o reaches a maximum at $i = 0, 180^\circ$ and is reduced to zero at $i = 54^\circ 44'$.

4. Calculating Perturbations of the First Order

We will use an orbit with the following elements as an example of calculating perturbations of the first order:

$$a = 7286.88 \text{ км}, \quad e = 0.099493, \quad i = 65^\circ.4900.$$

1. (Expressions for the coefficients Ω' and ω' were obtained in the work of D. Ye. Okhotsimskiy, T. M. Eneyev and G. P. Taratynova [1957]).

We will also assign the following values to the constants a' and J:

$$a' = 6378.39 \text{ км},$$
$$J = 0.00164147.$$

Substituting the adopted values of the orbital elements and constants a' and J in formulas (9), we get the final expression for the perturbations of the first order produced by the Earth's ellipticity:

$$\begin{aligned}
\delta_1 a = &-0.446 \cos M - 0.066 \cos 2M - 0.010 \cos 3M - 0.001 \cos 4M - \\
&-0.376 \cos(M + 2\omega) + 7.401 \cos(2M + 2\omega) + \\
&+ 2.584 \cos(3M + 2\omega) + 0.624 \cos(4M + 2\omega) + \\
&+ 0.131 \cos(5M + 2\omega) + 0.024 \cos(6M + 2\omega),
\end{aligned}$$

$$\begin{aligned}
\delta_1 e = &-0.0003046 \cos M - 0.0000454 \cos 2M - 0.0000066 \cos 3M - \\
&-0.0000002 \cos 4M + 0.0002601 \cos(M + 2\omega) + \\
&+ 0.0000003 \cos(M - 2\omega) - 0.0000251 \cos(2M + 2\omega) + \\
&+ 0.0005824 \cos(3M + 2\omega) + 0.0002119 \cos(4M + 2\omega) + \\
&+ 0.0000524 \cos(5M + 2\omega) + 0.0000113 \cos(6M + 2\omega) + \\
&+ 0.0000020 \cos(7M + 2\omega),
\end{aligned}$$

$$\begin{aligned}
\delta_1 i = &- 0^{\circ}.00136 \cos(M + 2\omega) + 0^{\circ}.01333 \cos(2M + 2\omega) + \\
&+ 0^{\circ}.00310 \cos(3M + 2\omega) + 0^{\circ}.00562 \cos(4M + 2\omega) + \\
&+ 0^{\circ}.00009 \cos(5M + 2\omega) + 0^{\circ}.00001 \cos(6M + 2\omega),
\end{aligned}$$

$$\begin{aligned}
\delta_1 \Omega = &-2^{\circ}.67416t - 0^{\circ}.00907 \sin M - 0^{\circ}.00067 \sin 2M - \\
&- 0^{\circ}.00007 \sin 3M - 0^{\circ}.00001 \sin 4M - 0^{\circ}.00149 \sin(M + 2\omega) + \\
&+ 0^{\circ}.01465 \sin(2M + 2\omega) + 0^{\circ}.00341 \sin(3M + 2\omega) + \\
&+ 0^{\circ}.00062 \sin(4M + 2\omega) + 0^{\circ}.00010 \sin(5M + 2\omega) + \\
&+ 0^{\circ}.00002 \sin(6M + 2\omega),
\end{aligned}$$

$$\begin{aligned}
\delta_1 \varepsilon = &-4^{\circ}.77956t - 0^{\circ}.01625 \sin M - 0^{\circ}.00154 \sin 2M - \\
&-0^{\circ}.00016 \sin 3M - 0^{\circ}.00002 \sin 4M - 0^{\circ}.00894 \sin(M + 2\omega) + \\
&+ 0^{\circ}.09471 \sin(2M + 2\omega) + 0^{\circ}.02376 \sin(3M + 2\omega) + \\
&+ 0^{\circ}.00461 \sin(4M + 2\omega) + 0^{\circ}.00080 \sin(5M + 2\omega) + \\
&+ 0^{\circ}.00014 \sin(6M + 2\omega),
\end{aligned}$$

$$
\begin{aligned}
e\delta_1\pi = &-0^\circ\!.31079t - 0^\circ\!.01846 \sin M - 0^\circ\!.00267 \sin 2M - \\
&-0^\circ\!.00038 \sin 3M - 0^\circ\!.00005 \sin 4M - 0^\circ\!.00001 \sin 5M - \\
&-0^\circ\!.01488 \sin (M + 2\omega) - 0^\circ\!.00649 \sin (2M + 2\omega) + \\
&+ 0^\circ\!.00002 \sin (M - 2\omega) + 0^\circ\!.03259 \sin (3M + 2\omega) + \\
&+ 0^\circ\!.01203 \sin (4M + 2\omega) + 0^\circ\!.00299 \sin (5M + 2\omega) + \\
&+ 0^\circ\!.00062 \sin (6M + 2\omega) + 0^\circ\!.00012 \sin (7M + 2\omega) + \\
&+ 0^\circ\!.00002 \sin (8M + 2\omega).
\end{aligned}
$$

Here the coefficients of the periodic perturbations of the large semiaxis are given in kilometers, but in the secular perturbations of the elements Ω, $\mathcal{E}$, π, the time should be presented in mean solar days.

The above-cited numerical values provide a clear idea of their magnitude.

Thus in the large semiaxis, the periodic perturbations with the arguments ($2M + 2\omega$) and ($3M + 2\omega$) are the most important, and their amplitudes are 7.4 and 2.6 kilometers. The greatest perturbation in the eccentricity has an argument ($3M + 2\omega$) and can produce a 4.2 deviation from the perigee altitude.

The greatest periodic perturbations in the other elements amount to several minutes of the arc. Thus there are fairly large perturbations in all the elements which should be taken into consideration.

5. Secular Perturbations of the Second Order in the Motion of a Node

The secular motion of a node is the most important feature of the motion of artificial satellites, as it can be very accurately determined from observations, on the one hand, and can be used for a more precise definition of the contraction of the Earth, on the other. The theoretical expression for the secular motion of a node should therefore be known as thoroughly as possible. Below is a calculation of the second order perturbations in relation to contraction in the motion of a node.

The longitude of an ascending node is defined by the following formula

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}} \cdot \frac{\partial R}{\partial i}, \tag{14}$$

where R is defined by formula (7).

We will designate the right part of the formula for Ω as Q and present it as

$$Q(a, e, i, \omega, M) = Na^{-7/2}\sqrt{1-\lambda^2}\Big[-1-3e\cos M+ + \cos(2M+2\omega) - \frac{1}{2}e\cos(M+2\omega) + \frac{7}{2}e\cos(3M+2\omega)\Big], \tag{15}$$

where $N = f\sqrt{fm}\,a'^2$ is the coefficient depending on the constants connected with the Earth.

In (15) we limited ourselves to the first power of eccentricity in order to obtain the secular perturbations, correct to the zero power of eccentricity.

Expanding the right part of the equation for Ω by degree of perturbation of the elements, we find that

$$\frac{d\Omega}{dt} = Q(a, e, i, \omega, M)_1 = Q + \frac{\partial Q}{\partial a}\delta_1 a + \frac{\partial Q}{\partial e}\delta_1 e + \frac{\partial Q}{\partial i}\delta_1 i + \frac{\partial Q}{\partial \omega}\delta_1\omega + \frac{\partial Q}{\partial M}\delta_1 M. \tag{16}$$

The unperturbed values of the elements should be represented in the right part of (16) in Q, as well as in its derivatives. After appropriate transformations and integration, the right part (with the exception of the term Q which, when integrated, produces δ_1, Ω) will show the perturbations of the second order in relation to the oblateness in Ω.

Since (9) does not contain any expressions for $\delta\omega$ perturbations, we will make use of the following known relationships

$$\left.\begin{aligned} &\delta M = \delta\varepsilon - \delta\pi - \frac{3}{2}\frac{n}{a}\int\delta a\,dt, \\ &\frac{\partial Q}{\partial\Omega} = -\frac{\partial Q}{\partial\omega}, \quad \frac{\partial Q}{\partial\varepsilon} = \frac{\partial Q}{\partial M}, \\ &\omega = \pi - \Omega, \quad \frac{\partial Q}{\partial\pi} = -\left(\frac{\partial Q}{\partial M} + \frac{\partial Q}{\partial\Omega}\right). \end{aligned}\right\} \tag{17}$$

Assuming that $\Omega = \Omega_0 + \delta_1\Omega + \delta_2\Omega + \cdots$, where Ω_0 is a constant and $\delta_i\Omega$ perturbations of the i-th order, we will find from (16), bearing (17) in mind, that

$$\delta_2\Omega = \int\left[\frac{\partial Q}{\partial a}\delta_1 a + \frac{\partial Q}{\partial e}\delta_1 e + \frac{\partial Q}{\partial i}\delta_1 i + \frac{\partial Q}{\partial \Omega}\delta_1\Omega + \frac{\partial Q}{\partial \varepsilon}\delta_1\varepsilon + \frac{\partial Q}{\partial \pi}\delta_1\pi - \frac{3}{2}\frac{n}{a}\frac{\partial Q}{\partial M}\int\delta_1 a dt\right]dt.$$

The calculation of individual integrals produces the following secular terms:

$$\int\frac{\partial Q}{\partial a}\delta_1 a dt = -\frac{7}{4}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\,nt,$$

$$\int\frac{\partial Q}{\partial i}\delta_1 i dt = -\frac{1}{4}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\,nt,$$

$$\int\frac{\partial Q}{\partial \Omega}\delta_1\Omega dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4(1-\lambda^2)\,nt,$$

$$\int\frac{\partial Q}{\partial e}\delta_1 e dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4\sqrt{1-\lambda^2}\left(1-\frac{3}{2}\lambda^2\right)\left(-3-\frac{1}{2}\cos 2\omega\right)nt +$$
$$+\frac{1}{8}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\left(-3\cos 2\omega+\frac{23}{3}\right)nt,$$

$$\int\frac{\partial Q}{\partial \pi}\delta_1\pi dt = \frac{1}{2}J^2\left(\frac{a'}{a}\right)^4\sqrt{1-\lambda^2}\left(1-\frac{3}{2}\lambda^2\right)\left(-3+\frac{1}{2}\cos 2\omega\right)nt -$$
$$-\frac{1}{8}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\left(-3\cos 2\omega-\frac{23}{3}\right)nt,$$

$$\int\frac{\partial Q}{\partial \varepsilon}\delta_1\varepsilon dt = -\frac{3}{2}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\,nt -$$
$$-\frac{1}{2}J^2\left(\frac{a'}{a}\right)^4\sqrt{1-\lambda^2}\left(\sqrt{1-\lambda^2}-1+\lambda^2\right)nt,$$

$$\int\left[\frac{3}{2}\cdot\frac{n}{a}\cdot\frac{\partial Q}{\partial M}\int\delta_1 a dt\right]dt = \frac{3}{4}J^2\left(\frac{a'}{a}\right)^4\lambda^2\sqrt{1-\lambda^2}\,nt.$$

Here we find the following expression for the coefficient of the secular perturbation in the motion of a node of the order of the square coefficient of J:

$$\Omega'_{J^2} = J^2\left(\frac{a'}{a}\right)^4\sqrt{1-\lambda^2}\left(\frac{19}{6}\lambda^2-\frac{5}{2}\right)n. \qquad (18)$$

It is not difficult to obtain also the coefficient of the secular perturbation in the longitude of a node of the first power in relation to magnitude D. Here we should use the fourth equation of system (8), and take the following expression as perturbation function

$$R = \frac{1}{35}D\frac{fma'^4}{r^5}(35\sin^4\delta - 30\sin^2\delta + 3).$$

The calculations produce the following expression for the secular motion of a node occasioned by this term of the perturbation function:

$$\Omega'_D = D\left(\frac{a'}{a}\right)\sqrt{1-\lambda^2}\left(\frac{3}{2}\lambda^2 - \frac{6}{7}\right)n. \tag{19}$$

Summing up Ω'_{J}, Ω'_{J^2} and Ω'_{D} [1], we will get the full secular motion of an ascending node that would take into account the terms of the second order in relations to the contraction of the Earth[2]

$$\Omega' = -J\left(\frac{a'}{a}\right)^2 \frac{\cos i}{(1-e^2)^2} n + J^2\left(\frac{a'}{c}\right)^4 \cos i\left(\frac{19}{6}\sin^2 i - \frac{5}{2}\right)n + $$
$$+ D\left(\frac{a'}{a}\right)^4 \cos i\left(\frac{3}{2}\sin^2 i - \frac{6}{7}\right)n, \tag{20}$$

where the terms with the coefficients J^2 and D are accurate only to the first power of orbital eccentricity.

Expression (20) enables us to the ellipticity of the terrestrial spheroid if the observed motion of the node is known.

6. Determining the Earth's Oblateness by the Observed Motion of a Node

To cite an example of the application of the above-developed theoretical expression of the motion of a node, we will calculate the oblateness of the Earth by using the elements of the second Soviet earth satellite mentioned in the work of King-Hele and Merson (1958), and the observed magnitude of the diurnal motion of the node to epoch 4.0 of January 1958 as indicated by King-Hele (1958). Since the elements of the January 4.0 datum are missing, we will define them by an interpolation between the two systems of elements nearest to that datum. We will get the following initial data:

Epoch 1958 January 4.0

P = 100.503m (draconic period)
a = 7161.19 km
e = 0.0802
i = 69.29°
Ω = 2.814°

---

1. Here for purposes of uniformity, the coefficient of the secular perturbation of the longitude of a first order node in relation to J is designated as Ω'_J.
2. A similar formula was developed in King-Hele's work [1958] but it was non-osculating elements.

We will assume the large semiaxis of the terrestrial ellipsoid a' to be 6,378.10 kilometers. To calculate the terms of the second order, where the approximate values of the J and D coefficients would be sufficient, we will assume that

$$J = 0.001637, \quad D = 0.0000106.$$

We will define the average motion n included in formula (20) by the following formula

$$n = \frac{360^\circ 1440^m}{P}. \tag{21}$$

Strictly speaking, the sidereal rather than the draconic period should be used in formula (21), but the error involved in this is so insignificant that it may be disregarded.

After the substitution of all these data in (20), we will get the following coefficient of J:

$$J = 0.001628.$$

The oblateness of the Earth ε we will find by the following formula

$$\varepsilon = J + \frac{m}{2} + \frac{\varepsilon^2}{2} - \frac{\varepsilon m}{7},$$

assuming that $m = 3449.79 \times 10^{-6}$. In this case, the reciprocal of the Earth's oblateness is equal to

$$\frac{1}{\varepsilon} = 297.9.$$

This value should be somewhat increased by a more thorough calculation of the average motion.

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